Class Exercise 3 Solution

1. Let D be the region bounded between $y = 6 - x^2, y = 2$, and $y = 0$. Express the double integral of some f over D in $dxdy$, $dydx$ and in polar coordinates.

   Solution. The parabola $y = 6 - x^2$ cuts the horizontal line $y = 2$ at $(2, 2)$ and $(-2, 2)$. It cuts the horizontal line $y = 0$ at $(0, \sqrt{6})$ and $(0, \sqrt{6})$. A ray from the origin cuts the parabola for $\theta \in [0, \pi/4]$ and $[3\pi/4, \pi]$. It cuts the line $y = 2$ for $\theta \in [\pi/4, 3\pi/4]$. We have

$$\iint_D f(x, y)\, dA = \int_0^2 \int_{-\sqrt{6-y}}^{\sqrt{6-y}} f(x, y)\, dxdy \ .$$

   Finally, in polar coordinates,

$$\begin{aligned}\iint_D f(x, y)\, dA &= \int_0^{\pi/4} \int_0^{r(\theta)} f(r\cos\theta, r\sin\theta) r\, drd\theta + \int_{\pi/4}^{3\pi/4} \int_0^{2/\sin\theta} f(r\cos\theta, r\sin\theta) r\, drd\theta \\ &\quad + \int_{3\pi/4}^{\pi} \int_0^{r(\theta)} f(r\cos\theta, r\sin\theta) r\, drd\theta,\end{aligned}$$

   where $r(\theta)$ is the polar equation of $y = 6 - x^2$, that is,

$$r(\theta) = \frac{-\sin\theta + \sqrt{\sin^2\theta + 24\cos^2\theta}}{2\cos^2\theta} \ .$$

   BTW, we also have

$$\iint_D f(x, y)\, dA = \int_{-\sqrt{6}}^{-2} \int_0^{6-x^2} f(x, y)\, dydx + \int_{-2}^{2} \int_0^{2} f(x, y)\, dydx + \int_2^{\sqrt{6}} \int_0^{6-x^2} f(x, y)\, dydx \ .$$

2. Find the area of one leaf of $r = 12\cos 3\theta$. Can you express this curve in cartesian coordinates?

   Solution. This rose has three leaves, one of which lies over $\theta \in [-\pi/6, \pi/6]$. The area is given by

$$\int_{-\pi/6}^{\pi/6} \int_0^{12\cos 3\theta} r\, drd\theta = 72 \int_{-\pi/6}^{\pi/6} \cos^2 3\theta\, d\theta = 12\pi.$$

   Using $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, the curve $r = 12\cos 3\theta$ becomes

$$r = 12\left(\frac{4x^3}{r^3} - 3\frac{x}{r}\right) \ ,$$

   which is simplified to

$$(x^2 + y^2)^2 = 48x^3 - 36x(x^2 + y^2) = 12x^3 - 36xy^2 \ .$$